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# Constructive Polynomial Approximation on the Sphere

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This paper considers the problem of constructive approximation of a continuous function on the unit sphere $S^{r-1} \subseteq \mathbb{R}^r$ by a spherical polynomial from the space $\mathbb{P}_n$ of all spherical polynomials of degree $\leqslant n$. In particular, for $r=3$ it is shown that the hyperinterpolation approximation $L_n f$ (in which the Fourier coefficients in the exact $L_2$ orthogonal projection $P_n f$ are approximated by a positive weight quadrature rule that integrates exactly all polynomials of degree $\leqslant 2n$) has the exact order $\|L_n\| \asymp n^{1/2}$ for its uniform norm, provided the underlying quadrature rule satisfies an additional "quadrature regularity" assumption. For $r=3$, this rate of growth is the same as that of $\|P_n\|$, and is known to be optimal among all linear projections on $\mathbb{P}_n$. For $r \geqslant 3$ an upper bound on $\|L_n\|$ of non-optimal asymptotic order $O(n^{(r-1)/2})$ also holds, without any special assumption on the quadrature rule. 

## 1. INTRODUCTION

In this paper we consider polynomial approximations on the unit sphere $S^{r-1} \subseteq \mathbb{R}^r$ from the space of all spherical polynomials of degree at most $n$ (i.e. the space of all polynomials in $r$ variables restricted to $S^{r-1}$).

In particular, we shall show for $r=3$ that the hyperinterpolation approximation introduced in [17] can have the optimal order of growth for its operator norm among all linear projections considered as maps from $C(S^{r-1})$ to $C(S^{r-1})$, namely $O(n^{1/2})$. The hyperinterpolation approximation $L_n f$ may be described as an approximation obtained from the partial sum of the Laplace (or Fourier) series for $f$, when the exact integrals in the $L_2$ inner products are approximated by a suitable quadrature rule: specifically, the quadrature rule must have positive weights, and must give the exact integral when applied to any polynomial of degree less than or equal to $2n$. A formal description of the hyperinterpolation approximation is given in Section 3. Examples of suitable quadrature rules are considered in Section 6. The operator norm $\|L_n\|_{C \to C}$ of $L_n$ as a map from $C(S^{r-1})$ to $C(S^{r-1})$ is studied in Section 5, and for $r=3$ shown there to be bounded



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by $n+1$ without further assumptions, and by $cn^{1/2}$ under a mild additional assumption on the quadrature rule.

The most studied polynomial approximation that needs only a finite number of point values of $f$ is the polynomial interpolant $\Lambda_n f$. This coincides with a given continuous function $f$ at a prescribed set of points $\{x_1, ..., x_{d_n}\} \subseteq S^{r-1}$, where $d_n \equiv d_n^{(r)}$ is the dimension of the space of spherical polynomials of degree at most $n$. Through the work of Reimer [16] and others, much is known about the norm $\|\Lambda_n\|_{C\to C}$ of $\Lambda_n$ as a map from $C(S^{r-1})$ to $C(S^{r-1})$, yet the problem of finding a set of interpolation points that yields a good uniform approximation, or of understanding how good such approximations can be, remains elusive. A bound on $\|\Lambda_n\|_{C\to C}$ given by Reimer [16] has, for $r=3$, the form $(n+1)(\lambda_{\text{avg}}/\lambda_{\min})^{1/2}$ (see Section 7), where $\lambda_{\text{avg}}$ and $\lambda_{\min}$ are the average and minimum eigenvalues of a certain positive-definite matrix. The ratio $\lambda_{\text{avg}}/\lambda_{\min}$ depends on the choice of points $\{x_1, ..., x_{d_n}\}$, but beyond the fact that $\lambda_{\text{avg}}/\lambda_{\min} \geqslant 1$ and the less obvious fact (shown by Reimer [16]) that $\lambda_{\text{avg}}/\lambda_{\min} > 1$ for $r \geqslant 3$ and $n \geqslant 3$, little seems to be known about its possible dependence on $r$ and $n$. One known result (see Section 7) is that for $r=3$ there exist interpolation points $\{x_1, ..., x_{d_n}\}$ (namely the “extremal fundamental systems” of Reimer [16]) such that $\|\Lambda_n\|_{C\to C} \leqslant (n+1)^2$. However, this result is almost certainly very pessimistic.

The simpler problem of the approximation properties of the hyperinterpolation operator $L_n$ as a map from $C(S^{r-1})$ to $L_2(S^{r-1})$ was studied in [17]. In that setting the approximation properties of $L_n$ are in a certain sense ideal, in that the norm of $L_n$ is shown in [17] to be given by

$$\|L_n\|_{C\to L_2} = |S^{r-1}|^{1/2}, \tag{1.1}$$

where $|S^{r-1}|$ denotes the surface area of the unit sphere. This is the best possible result, as is easily seen by considering the operator applied to the constant function 1. In contrast, it has been shown in [18] that for $r \geqslant 3$ and $n \geqslant 3$ the interpolation operator $\Lambda_n$ necessarily has a larger norm in the $C$ to $L_2$ sense, that is

$$\|\Lambda_n\|_{C\to L_2} > |S^{r-1}|^{1/2} \qquad \text{if } r \geqslant 3 \quad \text{and} \quad n \geqslant 3. \tag{1.2}$$

The proof of the latter result in [18] is by contradiction, and therefore gives no insight into the extent to which the inequality in (1.2) departs from equality.

The present paper, concentrating on the $C(S^{r-1})$ to $C(S^{r-1})$ setting, extends the known theoretical results for hyperinterpolation.

The results of computational experiments for the two approximation schemes will be published elsewhere.

Generic constants are denoted by $c$, while more specific constants are denoted $c_1$, $c_2$, $a_1$, $a_2$, etc.

## 2. PRELIMINARIES

For given $n \geqslant 0$, let $\mathbb{P}_n \equiv \mathbb{P}_n^{(r)}$ be the set of spherical polynomials of degree $\leqslant n$ in $r$ variables; i.e. the set of all polynomials in $r$ variables of degree at most $n$ restricted to $S^{r-1}$, the unit sphere in $\mathbb{R}^r$.

A popular basis for $\mathbb{P}_n^{(r)}$ is the set of spherical harmonics [14]

$$\{ Y_{\ell,k}^{(r)} : 1 \leqslant k \leqslant N(r,\ell),\ 0 \leqslant \ell \leqslant n \},$$

where

$$N(r,0) = 1, \quad N(r,\ell) = \frac{2\ell + r - 2}{\ell} \binom{\ell + r - 3}{\ell - 1} \qquad \text{for} \quad \ell \geqslant 1.$$

We shall assume that the spherical harmonics are normalized so that

$$\int_{S^{r-1}} Y_{\ell,k}^{(r)}(x)\, Y_{\ell',k'}^{(r)}(x)\, dx = \delta_{\ell\ell'} \delta_{kk'},$$

where $dx$ denotes surface measure on $S^{r-1}$. The dimension of the space $\mathbb{P}_n^{(r)}$ we denote by

$$d_n \equiv d_n^{(r)} = \sum_{\ell=0}^{n} N(r,\ell) = N(r+1,n). \tag{2.1}$$

For example, in the important special case $r=3$ we have $N(3,\ell) = 2\ell + 1$ and $d_n = (n+1)^2$.

The addition theorem of spherical harmonics [14] will play an important role. It states

$$\sum_{k=1}^{N(r,\ell)} Y_{\ell,k}^{(r)}(x)\, Y_{\ell,k}^{(r)}(y) = \frac{N(r,\ell)}{|S^{r-1}|} P_\ell^{(r)}(x \cdot y), \tag{2.2}$$

where $x \cdot y$ is the inner product in $\mathbb{R}^r$, $|S^{r-1}|$ is the surface area of the unit sphere,

$$|S^{r-1}| = \frac{r\pi^{r/2}}{\Gamma(1 + r/2)},$$

and $P_\ell^{(r)}$ is the Legendre polynomial of degree $\ell$ in $r$ dimensions, normalized by $P_\ell^{(r)}(1) = 1$.

## 3. THE APPROXIMATIONS DEFINED

For given dimension $r$ and degree $n$, let $X \equiv X_n^{(r)} = \{x_1, ..., x_{d_n}\} \subseteq S^{r-1}$ be a "fundamental system" of points on the sphere, meaning that the only member of $\mathbb{P}_n^{(r)}$ that vanishes at every point $x_j$, $j = 1, ..., d_n$, is the zero polynomial.

Given an arbitrary $f \in C(S^{r-1})$, we denote by $\Lambda_n f$ the unique polynomial in $\mathbb{P}_n$ that interpolates $f$ at each point of the fundamental system, that is

$$\Lambda_n f \in \mathbb{P}_n^{(r)}, \qquad \Lambda_n f(x_j) = f(x_j), \qquad j = 1, ..., d_n. \tag{3.1}$$

As a prelude to the introduction of the hyperinterpolation approximation, it is convenient to introduce an intermediate approximation, which is theoretically simpler but harder to compute, namely the $L_2$ orthogonal projection of $f$ onto $\mathbb{P}_n^{(r)}$, given by

$$P_n f = \sum_{\ell=0}^{n} \sum_{k=1}^{N(r,\ell)} (f, Y_{\ell,k}^{(r)})\, Y_{\ell,k}^{(r)}, \tag{3.2}$$

where $(\cdot, \cdot)$ is the $L_2$ inner product on $S^{r-1}$,

$$(u, v) := \int_{S^{r-1}} u(x)\, v(x)\, dx.$$

The hyperinterpolation approximation $L_n f$ is obtained by approximating the inner product in the Definition (3.2) of $P_n f$ by a positive-weight quadrature rule with the property of integrating all spherical polynomials of degree $\leqslant 2n$ exactly. Thus the hyperinterpolation approximation has the form

$$L_n f = \sum_{\ell=0}^{n} \sum_{k=1}^{N(r,\ell)} (f, Y_{\ell,k}^{(r)})_m\, Y_{\ell,k}^{(r)}, \tag{3.3}$$

where $(\cdot, \cdot)_m$ is a discrete version of the inner product obtained by application of an $m$-point quadrature formula,

$$(u, v)_m := \sum_{j=1}^{m} w_j u(t_j)\, v(t_j),$$

and where the weights $w_j$ and points $t_j$ in the quadrature rule $Q$,

$$Qg := \sum_{j=1}^{m} w_j g(t_j) \approx \int_{S^{r-1}} g(x)\, dx, \tag{3.4}$$

must satisfy

$$w_j > 0, \qquad t_j \in S^{r-1}, \qquad j = 1, ..., m, \tag{3.5}$$

and

$$Qp = \int_{S^{r-1}} p(x)\, dx, \qquad \forall p \in \mathbb{P}_{2n}^{(r)}. \tag{3.6}$$

According to [17], it follows from the definition that $m \geqslant d_n$.

Note that the hyperinterpolation approximation $L_n f$ depends on the choice of quadrature rule $Q$, just as the interpolation approximation $\Lambda_n f$ depends in the choice of fundamental system $X$. The notation will usually not make this dependence explicit.

All three of the approximations described here are linear projections onto $\mathbb{P}_n^{(r)}$, in that

$$p \in \mathbb{P}_n^{(r)} \Rightarrow \Lambda_n p = P_n p = L_n p = p.$$

In the last case this follows by observing for $p \in \mathbb{P}_n^{(r)}$ that

$$L_n p = \sum_{\ell=0}^{n} \sum_{k=1}^{N(r,\ell)} (p, Y_{\ell,k}^{(r)})_m\, Y_{\ell,k}^{(r)} = \sum_{\ell=0}^{n} \sum_{k=1}^{N(r,\ell)} (p, Y_{\ell,k}^{(r)})\, Y_{\ell,k}^{(r)} = p,$$

where the second equality follows from the exactness of the quadrature rule for polynomials of degree $\leqslant 2n$, and the last from the fact that the sum is just the Laplace or Fourier series for the polynomial $p$.

## 4. CONSTRUCTING $\Lambda_n f$ AND $L_n f$

In this section we consider alternative formulas for constructing $\Lambda_n f$ and $L_n f$, given a fundamental system $X$ in the first case, and a quadrature rule $Q$ in the second. At the same time we shall be developing reproducing-kernel representations that will be needed in the later theoretical analysis.

The most obvious way to compute the interpolant $\Lambda_n f$ is to represent it as a linear combination of spherical harmonics,

$$\Lambda_n f = \sum_{\ell=0}^{n} \sum_{k=1}^{N(r,\ell)} a_{\ell,k}\, Y_{\ell,k}^{(r)}, \tag{4.1}$$

where the coefficients $a_{\ell,k}$ must satisfy, from the interpolating Property 3.1, the linear system

$$\sum_{\ell=0}^{n} \sum_{k=1}^{N(r,\ell)} Y_{\ell,k}^{(r)}(x_j)\, a_{\ell,k} = f(x_j), \qquad j = 1, ..., d_n. \tag{4.2}$$

The matrix $\{Y^{(r)}_{\ell,k}(x_j)\}$ in this linear system is not singular, because of the assumption that $X=\{x_1, ..., x_{d_n}\}$ is a fundamental system. While the matrix elements $Y^{(r)}_{\ell,k}(x_j)$ with fixed $x_j$ can be computed relatively efficiently by exploiting recurrence relations of the spherical harmonics, the time for computing $\Lambda_n f$ will often be dominated by the time needed to solve the dense linear system (4.2). In contrast, the hyperinterpolation approximation $L_n f$ is already represented as a linear combination of spherical harmonics by the Definition (3.3), and does not need the solution of a linear system.

We develop here alternative representations of $\Lambda_n f$ and $L_n f$ (see (4.10) and (4.13) below), which may sometimes be preferred in practice because of their simplicity; in particular, explicit computation of spherical harmonics is avoided in these formulas.

To this end, it is useful to introduce the kernel $G_n(\cdot, \cdot)=G^{(r)}_n(\cdot, \cdot)$, defined by

$$G_n(x, y) := \sum_{\ell=0}^{n} \sum_{k=1}^{N(r,\ell)} Y^{(r)}_{\ell,k}(x)\, Y^{(r)}_{\ell,k}(y), \qquad x, y \in S^{r-1}. \tag{4.3}$$

This is a “reproducing kernel” in $\mathbb{P}^{(r)}_n$, because of the following elementary but important property:

LEMMA 4.1 (Reimer [16])

$$(p, G_n(\cdot, x)) = p(x) \qquad \forall p \in \mathbb{P}^{(r)}_n.$$

*Proof.* For $p \in \mathbb{P}^{(r)}_n$, the Definition 4.3 gives

$$(p, G_n(\cdot, x)) = \sum_{\ell=0}^{n} \sum_{k=1}^{N(r,\ell)} (p, Y^{(r)}_{\ell,k})\, Y^{(r)}_{\ell,k}(x),$$

which is simply the Laplace series representation of the spherical polynomial $p(x)$. ∎

It will be important to us that $G_n(x, y)$ is easily computed. The principal simplification is that $G_n(x, y)$ is “bizonal;” that is, its value depends only on the inner product $x \cdot y$ of the unit vectors $x$ and $y$. This follows from the addition theorem for spherical harmonics (2.2), which yields

$$G_n(x, y) = \tilde{G}_n(x \cdot y), \qquad x, y \in S^{r-1}, \tag{4.4}$$

where

$$\tilde{G}_n(z) = \frac{1}{|S^{r-1}|} \sum_{\ell=0}^{n} N(r,\ell)\, P^{(r)}_\ell(z), \qquad z \in [-1, 1]. \tag{4.5}$$

For example, if $r=3$ we have

$$\tilde{G}_n(z)=\frac{1}{4\pi}\sum_{\ell=0}^{n}(2\ell+1)\,P_\ell(z), \tag{4.6}$$

where $P_\ell(\,\cdot\,)$ is the usual Legendre polynomial. This $r=3$ result can be written in closed form, as pointed out by [10], in terms of the Jacobi polynomial $P_n^{(1,0)}$ (in the notation of Szegö [20]) appropriate to the weight function $(1-z)$. The closed form $r=3$ result (using [20, Equation (4.5.3)]) is

$$\tilde{G}_n(z)=\frac{n+1}{4\pi}\,P_n^{(1,0)}(z). \tag{4.7}$$

Of particular interest to us will be the value of $\tilde{G}_n(1)$, i.e. the value of $G_n(x, y)$ when $y=x$. According to (4.5) and (2.1) it is given by, as pointed out by Reimer [16],

$$\tilde{G}_n(1)=\frac{d_n}{|S^{r-1}|}. \tag{4.8}$$

For example, for $r=3$ it has the value

$$\tilde{G}_n(1)=\frac{(n+1)^2}{4\pi}.$$

To each point $x_j$ of the fundamental system $X=\{x_1, ..., x_{d_n}\}$ we may define a "kernel polynomial" $g_j\in\mathbb{P}_n^{(r)}$, by

$$g_j(x):=G_n(x, x_j)=\tilde{G}_n(x\cdot x_j), \qquad j=1, ..., d_n. \tag{4.9}$$

We shall say that $g_j$ is the kernel polynomial with axis $x_j$. It is easy to see that the set $\{g_1, ..., g_{d_n}\}\subseteq\mathbb{P}_n^{(r)}$ is linearly independent, because of the assumption that $X$ is a fundamental system, thus this set spans $\mathbb{P}_n^{(r)}$. Therefore the interpolating polynomial $\Lambda_n f$ may now, if we wish, be expressed in the form

$$\Lambda_n f=\sum_{j=1}^{d_n} e_j g_j, \tag{4.10}$$

where the real coefficients $e_j$ are determined by the linear system

$$\sum_{j=1}^{d} G_{ij}e_j=f(x_i), \qquad i=1, ..., d_n, \tag{4.11}$$

with

$$G_{ij} := g_j(x_i) = \tilde{G}_n(x_i \cdot x_j) = G_n(x_i, x_j), \qquad i, j = 1, ..., d_n. \tag{4.12}$$

The computation of the interpolant $\Lambda_n f$ via (4.10), (4.11), and (4.12), is easy to implement, requiring only the repeated evaluation of the polynomial $\tilde{G}_n$ and the solution of a linear system for $r = 3$. The time for computing the matrix $(G_{ij})$ is of order $O(nd_n^2) = O(n^5)$, and the time for a single evaluation of $\Lambda_n f(x)$ is of order $O(nd_n) = O(n^3)$, which in practice will be unimportant compared with the $O(d_n^3) = O(n^6)$ time needed to solve the linear system, unless the interpolant is required at very many points.

Now we describe an analogous expression for the hyperinterpolation approximation $L_n f$, obtained by interchanging the order of summation in (3.3). In this way we obtain

$$L_n f = \sum_{j=1}^{m} w_j f(t_j)\, g_j, \tag{4.13}$$

where this time $g_j$ denotes the kernel polynomial with axis $t_j$, that is

$$g_j(x) := G_n(x, t_j) = \tilde{G}_n(x \cdot t_j), \qquad j = 1, ..., m. \tag{4.14}$$

(It will be clear from the context whether $g_j$ has as its axis the point $x_j$ of the fundamental system $X$, or the point $t_j$ of the quadrature rule $Q$.)

We observe that (4.13) has a particularly simple structure, similar to the formula (4.10) for $\Lambda_n f$, but requiring no solution of a linear system.

## 5. HYPERINTERPOLATION IN THE UNIFORM NORM

In this section we study the hyperinterpolation operator $L_n$ as a map from $C(S^{r-1})$ to $C(S^{r-1})$.

Because $L_n$ is a linear projection on $\mathbb{P}_n$ we are able to argue in a standard way, that

$$\|L_n f - f\|_\infty = \|L_n(f - \chi) - (f - \chi)\|_\infty$$

for $\chi$ an arbitrary polynomial in $\mathbb{P}_n$. From this it follows immediately that

$$\|L_n f - f\|_\infty \leqslant (1 + \|L_n\|_{C \to C})\, E_n(f), \tag{5.1}$$

where

$$\|L_n\|_{C \to C} = \sup_{f \in C,\, f \neq 0} \frac{\|L_n f\|_\infty}{\|f\|_\infty},$$

and $E_n(f)$ is the error of best uniform approximation,

$$E_n(f) = \inf_{\chi \in \mathbb{P}_n} \|f - \chi\|_\infty.$$

Thus our task reduces, in the usual way, to the study of the norm of the operator $L_n$ in the setting $C$ to $C$.

To guide us in assessing the quality of $L_n$, it is useful to recall first that $P_n$, the $L_2$ orthogonal projection, is the minimal norm projection in the setting $C$ to $C$: that is, if $\Omega$ is an arbitrary linear projection onto $\mathbb{P}_n^{(r)}$, then

$$\|P_n\|_{C \to C} \leqslant \|\Omega\|_{C \to C}.$$

This result was proved by Berman [2] for the case $r = 2$, and extended to general $r$ by Daugavet [5]; a proof for $r \geqslant 3$ is given by Reimer [16]. Moreover, for $r = 2$ it is known (see [6]) that

$$\|P_n\|_{C \to C} \asymp \log n,$$

while for $r = 3$ it was shown by [11] that

$$\|P_n\|_{C \to C} \asymp n^{1/2}, \tag{5.2}$$

where $a_n \asymp b_n$ means that there exist positive constants $c_1$ and $c_2$ such that $c_1 a_n \leqslant b_n \leqslant c_2 a_n$. The generalization of this result to arbitrary $r \geqslant 3$, discussed by Reimer [16, Section 11], is

$$\|P_n\|_{C \to C} \asymp n^{(r-2)/2}. \tag{5.3}$$

There are two main results in this section. First, in Theorem 5.5.2 we establish a general result, that $\|L_n\|_{C \to C}$ is bounded above by $d_n^{1/2}$. For the important special case $r = 3$ Theorem 5.5.2 yields

$$\|L_n\|_{C \to C} \leqslant n + 1.$$

This is an improvement on the result $\|L_n\|_{C \to C} \leqslant cn^2$ obtained by [10, Theorem 3.2(i)]. On the other hand, for all $r \geqslant 3$ the rate of growth of $d_n^{1/2}$ with $n$, namely $O(n^{(r-1)/2})$, is worse by a factor of $n^{1/2}$ than the optimal result for $\|P_n\|_{C \to C}$ given by (5.3). That prompts the question of whether better results for $\|L_n\|_{C \to C}$ can be achieved.

In Theorem 5.5.4 we obtain, for the special case $r = 3$ and under a mild additional assumption on the quadrature rule, the improved result that $\|L_n\|_{C \to C} \asymp n^{1/2}$, which, as we have noted, is optimal with respect to order.

The following simple lemma provides the foundation for these results. In this lemma $g_j(x) = G(x, t_j)$ is the kernel polynomial with axis $t_j$, as in (4.14).

LEMMA 5.5.1. *The norm of the hyperinterpolation operator $L_n$ in the setting C to C is given by*

$$\|L_n\|_{C\to C} = \max_{x\in S^{r-1}} \sum_{j=1}^{m} w_j\,|g_j(x)|.$$

*Proof.* From the expression (4.13) for $L_n f$ we have

$$|L_n f(x)| \leqslant \sum_{j=1}^{m} w_j\,|f(t_j)|\,|g_j(x)| \leqslant \sum_{j=1}^{m} w_j\,|g_j(x)|\,\|f\|_\infty,$$

from which it follows that

$$\|L_n f\|_\infty \leqslant \max_{x\in S^{r-1}} \sum_{j=1}^{m} w_j\,|g_j(x)|\,\|f\|_\infty, \tag{5.4}$$

where we exploit here the continuity of the spherical polynomials $g_j$ for $j=1, ..., m$.

The proof is completed by showing that there exists $f^*\in C(S^{r-1})$, with $f^*\neq 0$, for which (5.4) is an equality. To this end, let $x_0\in S^{r-1}$ achieve the maximum in the sum in (5.4), i.e.

$$\sum_{j=1}^{m} w_j\,|g_j(x_0)| = \max_{x\in S^{r-1}} \sum_{j=1}^{m} w_j\,|g_j(x)|. \tag{5.5}$$

Then define $f^*\in C(S^{r-1})$ such that $\|f^*\|_\infty=1$ and

$$f^*(t_j) = \text{sign } g_j(x_0), \qquad j=1, ..., m;$$

concrete constructions of $f^*$ satisfying these conditions can be accomplished in well-known ways. By (4.13) we now have

$$L_n f^*(x) = \sum_{j=1}^{m} w_j f^*(t_j)\, g_j(x) = \sum_{j=1}^{m} w_j\, g_j(x) \text{ sign } g_j(x_0),$$

and hence, on setting $x=x_0$,

$$L_n f^*(x_0) = \sum_{j=1}^{m} w_j\,|g_j(x_0)| = \max_{x\in S^{r-1}} \sum_{j=1}^{m} w_j\,|g_j(x)| = \|L_n f^*\|_\infty$$

so that (5.4) is an equality for $f=f^*$. This completes the proof. ∎

With the aid of Lemma 5.5.1 we can now turn to estimating the rate of growth of $\|L_n\|_{C\to C}$ with $n$. The first result needs no further assumptions.

THEOREM 5.5.2. *The norm of the hyperinterpolation operator in the setting C to C is bounded by*

$$\|L_n\|_{C\to C} \leqslant d_n^{1/2}.$$

*Proof.* Let $x_0 \in S^{r-1}$ satisfy (5.5), as in the proof of Lemma 5.5.1. Then that lemma gives, by the use of (4.14) and the Cauchy–Schwarz inequality,

$$\begin{aligned} \|L_n\|_{C\to C} &= \sum_{j=1}^{m} w_j \, |g_j(x_0)| = \sum_{j=1}^{m} w_j \, |G_n(x_0, t_j)| \\ &= \sum_{j=1}^{m} w_j^{1/2} w_j^{1/2} \, |G_n(x_0, t_j)| \\ &\leqslant \left( \sum_{j=1}^{m} w_j \right)^{1/2} \left( \sum_{j=1}^{m} w_j G_n(x_0, t_j)^2 \right)^{1/2} \\ &= |S^{r-1}|^{1/2} \left( \int_{S^{r-1}} G_n(x_0, x)^2 \, dx \right)^{1/2}, \end{aligned}$$

where in the last step we twice used the property (3.6) of the quadrature rule $Q$. The reproducing kernel property in Lemma 4.1, combined with the expression (4.8) for the "polar" value of $\tilde{G}_n$, gives

$$\int_{S^{r-1}} G_n(x_0, x)^2 \, dx = G_n(x_0, x_0) = \frac{d_n}{|S^{r-1}|},$$

from which it follows that

$$\|L_n\|_{C\to C} \leqslant d_n^{1/2}. \quad \blacksquare$$

Now we set $r=3$. To obtain the promised more precise results we need to assume that the family of $m$-point quadrature rules $Q_m$ has, in addition to the property (3.6), also a certain regularity property: roughly speaking, that the contribution to the quadrature sum from a spherical cap of a reasonable size, when $Q_m$ is applied to the function $f \equiv 1$, is never unboundedly large in relation to the corresponding integral. A "spherical cap of spherical radius $\alpha$" is the closed cap cut from $S^2$ by a cone of half-angle $\alpha$; the "axis" of the spherical cap is the polar axis of the cone. The surface area of a spherical cap of spherical radius $\alpha$ is $2\pi(1-\cos\alpha) \approx \pi\alpha^2$ when $\alpha$ is small. From this it follows that the expected contribution from a spherical cap of spherical radius $1/\sqrt{m}$ is approximately $\pi(1/\sqrt{m})^2 = \pi/m$. In this light it is natural to require that the contribution from the quadrature points actually found in such a spherical cap be bounded by $c/m$, where $c$ is a constant. This motivates the following assumption, which we shall call the "quadrature regularity" assumption.

*Assumption* 1 (Quadrature regularity). Let $r = 3$. The infinite family of positive-weight $m$-point quadrature rules $\{Q_m\}$ is said to satisfy the quadrature regularity assumption if there exists $c_1 > 0$, with $c_1$ independent of $m$, such that for every spherical cap $A$ of spherical radius $1/\sqrt{m}$ we have

$$\sum_{t_j \in A} w_j \leqslant c_1 \, |A|, \tag{5.6}$$

where $|A| = 2\pi(1 - \cos(1/\sqrt{m})) \approx \pi/m$ is the surface area of $A$.

In Section 6 we show that many practical quadrature schemes do satisfy this assumption. It follows from the assumption, as we shall see in Lemma 5.5.3, that spherical caps with spherical radius larger than $1/\sqrt{m}$ display the same regularity property.

We shall also need an analogous regularity result for "spherical collars", where by a spherical collar we mean the difference between two spherical caps with the same axis. The "spherical radii" of a spherical collar which is the difference of spherical caps with spherical radii $\beta$ and $\alpha$, with $\beta > \alpha$, are $\alpha$ and $\beta$. The "height" of such a spherical collar is $\cos\alpha - \cos\beta$, and the "spherical height" is $\beta - \alpha$.

LEMMA 5.5.3. *Let $\{Q_m\}$ be an infinite family of positive-weight m-point quadrature rules on $S^2$ satisfying the quadrature regularity assumption. Then there exist constants $c_2, c_3 > 0$, independent of m, such that for every spherical cap $A_\alpha$ of spherical radius $\alpha \geqslant 1/\sqrt{m}$ we have*

$$\sum_{t_j \in A_\alpha} w_j \leqslant c_2 \, |A_\alpha|,$$

*and for every spherical collar $B_{\alpha,\beta}$ with spherical radii $\alpha$, $\beta$ with $0 < \alpha < \beta \leqslant \pi$ and $\beta - \alpha \geqslant 1/\sqrt{m}$ we have*

$$\sum_{t_j \in B_{\alpha,\beta}} w_j \leqslant c_3 \, |B_{\alpha,\beta}|.$$

*Proof.* Let $x_0 \in S^2$ be the axis of $A_\alpha$ and $B_{\alpha,\beta}$. We first prove the second result for the special case of a spherical collar $B_{\alpha,\beta}$ whose spherical height $\beta - \alpha$ is exactly equal to $1/\sqrt{m}$.

Let $\gamma = (\beta + \alpha)/2$, and consider the latitude $\ell_\gamma = \{x \in S^2 : x \cdot x_0 = \cos \gamma\}$, which is a circle on $S^2$ of radius $\sin \gamma$. We claim that (as pointed out to us by D. Mauersberger, private communication) $B_{\alpha, \beta}$ can be covered by $K = \lceil 2\pi \sqrt{m} \sin \gamma \rceil$ spherical caps $D_1, ..., D_K$ of spherical radius $1/\sqrt{m}$, with axes $x_1, ..., x_K$ equally spaced on the circle $\ell_\gamma$. To see this, note first that the distance between successive axes (measured along $\ell_\gamma$) is $2\pi \sin \gamma / K \leqslant 1/\sqrt{m}$. Given $x \in B_{\alpha, \beta}$, let $\tilde{x}$ denote the nearest point to $x$ on $\ell_\gamma$, and let $x_j$ be a spherical cap axis closest to $\tilde{x}$. The great-circle distance between $x$ and $x_j$ is bounded above by the sum of the distance between $x_j$ and $\tilde{x}$ along $\ell_\gamma$ and the great-circle distance between $\tilde{x}$ and $x$, and so is bounded above by $\frac{1}{2}(1/\sqrt{m}) + \frac{1}{2}(1/\sqrt{m}) = 1/\sqrt{m}$, thus $x \in D_j$. Thus the claim is established. It follows from this and the quadrature regularity assumption that

$$\sum_{t_j \in B_{\alpha, \beta}} w_j \leqslant K c_1 \, |D_1| \leqslant \frac{c_1 \pi K}{m} \leqslant \frac{c_1 \pi}{m} \, (2\pi \sqrt{m} \sin \gamma + 1),$$

where we have used $|D_1| = 2\pi(1 - \cos(1/\sqrt{m})) \leqslant \pi/m$. We also have

$$\begin{aligned} |B_{\alpha, \beta}| &= 2\pi(\cos \alpha - \cos \beta) \\ &= 4\pi \sin \gamma \sin(1/2 \sqrt{m}) \\ &\geqslant 4 \sin \gamma / \sqrt{m}, \end{aligned} \tag{5.7}$$

where in the last step we used $\sin \theta \geqslant 2\theta/\pi$ for $0 \leqslant \theta \leqslant \pi/2$. Putting the results together, we have

$$\sum_{t_j \in B_{\alpha, \beta}} w_j \leqslant c_1 \pi \left( \frac{\pi}{2} \, |B_{\alpha, \beta}| + \frac{1}{m} \right) \leqslant c_1 \, \frac{3\pi^2}{4} \, |B_{\alpha, \beta}| \,,$$

where in the last step we used (5.7) together with $\sin \gamma \geqslant \sin(1/2 \sqrt{m}) \geqslant 1/\pi \sqrt{m}$ to obtain $1/m = (\pi/\sqrt{m})(1/(\pi \sqrt{m})) \leqslant (\pi/4) \, |B_{\alpha, \beta}|$. Thus the spherical collar result is proved for $\beta - \alpha = 1/\sqrt{m}$ with $c_3 = c_1 3\pi^2/4$.

Now consider an arbitrary spherical collar $B_{\alpha, \beta}$ with $\beta - \alpha \geqslant 1/\sqrt{m}$. If $\beta - \alpha$ is an integer multiple of $1/\sqrt{m}$ the result extends trivially, with the same constant as above. Otherwise, following an argument due to D. Mauersberger (private communication), we consider the two sub-collars

$$B_\alpha = B_{\alpha, \alpha + k/\sqrt{m}} \,, \qquad B_\beta = B_{\beta - k/\sqrt{m}, \beta} \,,$$

where $k=\lfloor(\beta-\alpha)\sqrt{m}\rfloor$. Note that $B_\alpha \subseteq B_{\alpha,\beta}$ and $B_\beta \subseteq B_{\alpha,\beta}$, and that every point of $B_{\alpha,\beta}$ is in at least one of $B_\alpha$, $B_\beta$ because $k \geqslant 1$. Thus

$$\sum_{x_j \in B_{\alpha,\beta}} w_j \leqslant \sum_{x_j \in B_\alpha} w_j + \sum_{x_j \in B_\beta} w_j \leqslant c_1 \frac{3\pi^2}{4} (|B_\alpha| + |B_\beta|)$$
$$\leqslant \frac{c_1 3\pi^2}{2} |B_{\alpha,\beta}|.$$

Thus the spherical collar result is now proved, with $c_3 = c_1 3\pi^2/2$.

Finally, to prove the spherical cap result for a spherical cap $A_\alpha$ of spherical radius $\alpha \geqslant 1/\sqrt{m}$, let $A$ denote the spherical cap with the same axis and spherical radius $1/\sqrt{m}$, and let $B_\alpha$ be the spherical collar with the same axis and spherical radii $\alpha - k/\sqrt{m}$ and $\alpha$, where $k = \lceil \alpha\sqrt{m} \rceil - 1$. Both $A$ and $B_\alpha$ are subsets of $A_\alpha$, and every point of $A_\alpha$ is in at least one of $A$ and $B_\alpha$, thus we obtain

$$\sum_{x_j \in A_\alpha} w_j \leqslant \sum_{x_j \in A} w_j + \sum_{x_j \in B_\alpha} w_j$$
$$\leqslant c_1 |A| + c_1 \frac{3\pi^2}{4} |B_\alpha|$$
$$\leqslant c_1 \left(1 + \frac{3\pi^2}{4}\right) |A_\alpha|.$$

Thus the spherical collar result is proved, with $c_2 = c_1(1 + 3\pi^2/4)$. ▮

We now turn to the foreshadowed estimation of $\|L_n\|_{C \to C}$.

*Remark* 1. In the following theorem we do not need the property (3.6) of $Q_m$. While the theorem can be applied to families of rules $\{Q_m\}$ not satisfying this condition, it needs to be remembered that the estimate (5.1) for the error in $L_n f$ is no longer valid in this case.

*Remark* 2. The condition $m \geqslant (n+1)^2$ in the theorem below, relating $m$ (the number of quadrature points) with $n$ (the degree of the spherical polynomial space $\mathbb{P}_n^{(r)}$) is natural, in that it is shown in [17] to be necessary if (3.6) holds.

THEOREM 5.5.4. *Let $r = 3$. For each $n \geqslant 0$ let $m = m_n$ satisfy $m \geqslant (n+1)^2$. Moreover let $\{Q_m\}$ be a family of positive-weight m-point quadrature rules satisfying the quadrature regularity assumption. Then there exists $c_4 > 0$, with $c_4$ independent of n, such that*

$$\|L_n\|_{C \to C} \leqslant c_4 n^{1/2}, \tag{5.8}$$

*where* $L_n f \in \mathbb{P}_n^{(3)}$ *is the hyperinterpolation approximation associated with* $Q_m$.

*Proof.* Let $x_0$ satisfy (5.5), so that from Lemma 5.5.1, together with (4.14) and (4.7), we have, on setting $z_j = x_0 \cdot t_j$,

$$\|L_n\|_{C \to C} = \sum_{j=1}^{m} w_j \, |g_j(x_0)| = \frac{n+1}{4\pi} \sum_{j=1}^{m} w_j \, |P_n^{(1,0)}(z_j)|$$

$$\leqslant \frac{n+1}{4\pi} \sum_{z_j \geqslant 0} w_j (|P_n^{(1,0)}(z_j)| + |P_n^{(0,1)}(z_j)|),$$

where in the last step we have used $P_n^{(1,0)}(-z) = (-1)^n \, P_n^{(0,1)}(z)$ (see Szegö [20, Equation (4.1.3)]). Both $|P_n^{(1,0)}(z)|$ and $|P_n^{(0,1)}(z)|$ are bounded by $P_n^{(1,0)}(1) = n+1$, from [20, Equation (4.1.1) and the last sentence of page 168]. Together with Equation (7.32.6) of [20] this gives

$$|P_n^{(1,0)}(\cos\theta)| + |P_n^{(0,1)}(\cos\theta)| \leqslant \min(2(n+1), c_5 n^{-1/2} \theta^{-3/2}), \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}, \tag{5.9}$$

for some constant $c_5 > 0$.

We denote the right-hand side of this inequality by $u_n(\cos\theta)$. Thus

$$\|L_n\|_{C \to C} \leqslant \frac{n+1}{4\pi} \sum_{z_j \geqslant 0} w_j u_n(z_j), \tag{5.10}$$

where $u_n$ is monotone nondecreasing on $[0, 1]$. We split the right-hand side into a "main" term $M$, which contains the contributions to the sum for $0 \leqslant z_j < z_0$, and a remainder $R$, containing the contributions from the spherical cap $z \geqslant z_0$, with $z_0$ a number yet to be specified, but which satisfies $0 < z_0 \leqslant 1 - 1/n$.

The main term $M$ is to be handled by bounding it above by a Riemann sum, and thence by a 1-dimensional integral. Thus we partition the interval $[0, z_0]$ by defining

$$\xi_k = z_0 - \frac{N-k}{n} \qquad \text{for} \quad k = 1, ..., N,$$

where $N = \lceil n z_0 \rceil \leqslant n - 1$, so that

$$\xi_1 > 0, \ \xi_N = z_0, \qquad \text{and} \qquad \xi_{k+1} - \xi_k = \frac{1}{n} \qquad \text{for } k = 1, ..., N-1.$$

We also define $\xi_0 = \xi_1 - 1/n$ and $\xi_{N+1} = z_0 + 1/n \leqslant 1$. Now we can write

$$M = \frac{n+1}{4\pi} \sum_{0 \leqslant z_j < z_0} w_j u_n(z_j) = \frac{n+1}{4\pi} \sum_{k=0}^{N-1} \sum_{\xi_k \leqslant z_j < \xi_{k+1}} w_j u_n(z_j),$$

where for $z < 0$ we define $u_n(z) = 0$. Using the monotonicity of $u_n$, we then deduce

$$M \leqslant \frac{n+1}{4\pi} \sum_{k=0}^{N-1} \left( \sum_{\xi_k \leqslant z_j < \xi_{k+1}} w_j \right) u_n(\xi_{k+1}). \tag{5.11}$$

We may bound the sum of the weights over each spherical collar of height $1/n$ by appeal to Lemma 5.5.3. (Note that $1/n > 1/\sqrt{m}$, and that for $0 \leqslant \alpha < \beta \leqslant \pi$ it is clear that the spherical height $\beta - \alpha$ of the spherical collar is greater than the height $\cos\alpha - \cos\beta = 1/n$, from which it follows that $\beta - \alpha \geqslant 1/n > 1/\sqrt{m}$.) Since the surface area of a spherical collar of height $1/n$ is $2\pi/n$, Lemma 5.5.3 gives

$$\sum_{\xi_k \leqslant z_j < \xi_{k+1}} w_j \leqslant 2\pi c_3/n. \tag{5.12}$$

Moreover, using again the monotonicity of $u_n$ we obtain

$$\begin{aligned}
\frac{1}{n} \sum_{k=0}^{N-1} u_n(\xi_{k+1}) &\leqslant \sum_{k=0}^{N-1} \int_{\xi_{k+1}}^{\xi_{k+2}} u_n(\xi)\, d\xi = \int_{\xi_1}^{\xi_{N+1}} u_n(\xi)\, d\xi \\
&\leqslant \int_0^1 u_n(\xi)\, d\xi \leqslant c_5 n^{-1/2} \int_0^{\pi/2} \theta^{-3/2} \sin\theta\, d\theta \\
&\leqslant c_5 n^{-1/2} \int_0^{\pi/2} \theta^{-1/2}\, d\theta = \sqrt{2\pi} c_5 n^{-1/2}.
\end{aligned}$$

From this, together with (5.11) and (5.12), it follows that $M \leqslant \sqrt{2\pi} c_3 c_5 n^{1/2}$. Note that the sum is independent of $z_0$.

Now we turn to the term $R$, given by

$$R = \frac{n+1}{4\pi} \sum_{z_j \geqslant z_0} w_j u_n(z_j).$$

After preliminary study it turns out to be adequate to partition the region $z_j \geqslant z_0$ into a spherical cap and two spherical collars of appropriate sizes and all with axis $x_0$, and to estimate the contribution to $R$ from each using Lemma 5.5.3. Specifically, let $A_\kappa$ be the spherical cap with axis $x_0$ and with spherical radius $n^{-\kappa}$, where $0 < \kappa < 1$, let $B_\gamma$ be the spherical collar that adjoins $A_\kappa$ and has spherical height $n^{-\gamma}$, where $0 < \gamma \leqslant \kappa$, and finally let $D$

be the spherical collar that adjoins $B_\gamma$ and has height $n^{-1}$. The union $A_\kappa \cup B_\gamma \cup D$ is the intersection of $S^2$ with the half-space $z = \cos\theta \geqslant z_0$ if we choose

$$z_0 = \cos\left(\frac{1}{n^\kappa} + \frac{1}{n^\gamma}\right) - \frac{1}{n},$$

which we note ensures $z_0 \in (0, 1 - 1/n)$ if $n$ is sufficiently large. Introducing an obvious notation, we may write

$$R = R(A_\kappa) + R(B_\gamma) + R(D). \tag{5.13}$$

For the term $R(A_\kappa)$ the first part of Lemma 5.5.3 is applicable, because $n^{-\kappa} > n^{-1}$ (since $\kappa < 1$), which in turn exceeds $m^{-1/2}$, thus from the lemma and (5.9) we obtain

$$\begin{aligned} R(A_\kappa) &\leqslant \frac{n+1}{4\pi} c_2 \, |A_\kappa| \, u_n(1) \leqslant \frac{n+1}{4\pi} c_2 \, \pi (n^{-\kappa})^2 \, 2(n+1) \\ &= \frac{1}{2} c_2 (n+1)^2 \, n^{-2\kappa} \leqslant 2c_2 n^{2-2\kappa}. \end{aligned}$$

For the term $R(B_\gamma)$ we may again use Lemma 5.5.3, since the spherical height of the collar is $n^{-\gamma} > n^{-1} > m^{-1/2}$, thus

$$\begin{aligned} R(B_\gamma) &\leqslant \frac{n+1}{4\pi} c_3 \, |B_\gamma| \, u_n(\cos n^{-\kappa}) \\ &\leqslant \frac{n+1}{4\pi} c_3 \pi (n^{-\gamma} + n^{-\kappa})^2 \, c_5 n^{-1/2} (n^\kappa)^{3/2} \leqslant 2c_3 c_5 n^{1/2 - 2\gamma + 3\kappa/2}, \end{aligned}$$

where we used (5.9) and $n^{-\kappa} \leqslant n^{-\gamma}$ (since $0 < \gamma \leqslant \kappa$).

Finally, for the term $R(D)$ we have, similarly, since $|D| = 2\pi/n$,

$$\begin{aligned} R(D) &\leqslant \frac{n+1}{4\pi} c_3 \frac{2\pi}{n} u_n(\cos(n^{-\kappa} + n^{-\gamma})) \\ &\leqslant c_3 u_n(\cos(n^{-\gamma})) \leqslant c_3 c_5 n^{-1/2} n^{3\gamma/2}. \end{aligned}$$

From these three estimates and (5.13) it follows that $R \leqslant (2c_2 + 3c_3 c_5)\, n^{1/2}$, provided we can simultaneously satisfy

$$0 < \gamma \leqslant \kappa < 1, \; 2 - 2\kappa \leqslant \tfrac{1}{2}, \; \tfrac{1}{2} - 2\gamma + \tfrac{3}{2}\kappa \leqslant \tfrac{1}{2}, \; -\tfrac{1}{2} + \tfrac{3}{2}\gamma \leqslant \tfrac{1}{2}.$$

Since these are all satisfied by, for example,

$$\kappa = \tfrac{3}{4}, \qquad \gamma = \tfrac{2}{3},$$

it follows from this result and $M \leqslant \sqrt{2\pi} c_3 c_5 n^{1/2} < 3 c_3 c_5 n^{1/2}$ that

$$\|L_n\|_{C \to C} = M + R \leqslant (2c_2 + 6c_3 c_5)\, n^{1/2},$$

completing the proof. ∎

## 6. CONFORMING QUADRATURE SCHEMES FOR $r = 3$

In this section we show that many quadrature schemes possess the quadrature regularity property. We begin with a simple sufficient condition.

PROPOSITION 6.6.1. *An infinite family of m-point quadrature rules* $\{Q_m\}$ *with positive weights* $w_1, ..., w_m$ *and points* $x_1, .., x_m \in S^2$ *satisfies the quadrature regularity property if there exist* $a_1, a_2 > 0$, *independent of m, such that*

$$w_j \leqslant \frac{a_1}{m}, \qquad j = 1, ..., m \tag{6.1}$$

*and*

$$\sphericalangle (x_j, x_k) \geqslant \frac{a_2}{\sqrt{m}}, \qquad j, k = 1, ..., m, \qquad j \neq k. \tag{6.2}$$

*Proof.* For an arbitrary spherical cap $A_m$ of spherical radius $1/\sqrt{m}$, it follows readily from (6.2) that the number of points contained in $A_m$ is bounded independently of $m$. The quadrature regularity property then follows immediately from (6.1) on noting that $|A_m|$ is of exact order $1/m$.

One important class of quadrature rules does not in general satisfy the conditions of Proposition 6.6.1, but nevertheless can be quadrature regular. These are the tensor-product rules, which we may introduce this way. The surface integral of $f \in C(S^2)$ can be written as

$$\int_{S^{r-1}} f(x)\, dx = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_{-1}^{1} F(z, \phi)\, dz\, d\phi, \tag{6.3}$$

where $\phi$ is the azimuthal angle and $\theta$ the polar angle, $z = \cos\theta$, and $F(\cos\theta, \phi) = f(\theta, \phi)$. A tensor-product rule for the integral (6.3) is a rule of the form

$$\sum_k \mu_k \sum_j v_j F(z_j, \phi_k) = q_\phi q_z F, \tag{6.4}$$

where

$$q_\phi g := \sum_k \mu_k g(\phi_k), \qquad q_z h := \sum_j v_j h(z_j), \tag{6.5}$$

for appropriate choices of the 1-dimensional rules $q_\phi$ and $q_z$, and with both sums finite. For the azimuthal integration a sensible choice for the rule $q_\phi$, and the only one considered here, is the rectangle rule with spacing $\pi/(n+1)$,

$$q_\phi g := \frac{\pi}{n+1} \sum_{k=0}^{2n+1} g\left(\frac{k\pi}{n+1}\right), \tag{6.6}$$

because this rule is exact for all trigonometric polynomials of degree $\leqslant 2n+1$.

To see how the rule $q_z$ should be chosen, recall that the tensor-product rule $q_\phi q_z$ is required to be exact for all spherical polynomials of degree $\leqslant 2n$. Equivalently, we need the rule $q_\phi q_z$ to be exact if $f$ is an arbitrary spherical harmonic $Y_{\ell,k}$ of degree $\leqslant 2n$. The spherical harmonics can be chosen as

$$Y_{\ell,k}(\theta,\phi) = \begin{cases} c_{\ell m} P_\ell^m(\cos\theta)\cos m\phi & \text{for} \quad k = 2m+1, m = 0, ..., \ell \\ c_{\ell m} P_\ell^m(\cos\theta)\sin m\phi & \text{for} \quad k = 2m, m = 1, ..., \ell, \end{cases}$$

where $P_\ell^m$ is an associated Legendre function of the first kind. Our choice of azimuthal quadrature rule already ensures, because $m \leqslant \ell$, that $\cos m\phi$ and $\sin m\phi$ are integrated exactly for all $\ell \leqslant 2n$, and therefore ensures that

$$q_\phi Y_{\ell,k} = 0 \qquad \text{for} \quad 2 \leqslant k \leqslant 2\ell+1, 0 \leqslant \ell \leqslant 2n.$$

Thus the desired property holds for spherical harmonics $Y_{\ell,k}$ with $k > 1$, and it only remains to prove it for $k = 1$. Since $P_\ell^0 = P_\ell$, the Legendre polynomial, it follows that the property will hold if and only if

$$q_z h = \int_{-1}^{1} h(z)\, dz \qquad \forall h \in \mathbb{P}_{2n}[-1, 1]. \tag{6.7}$$

In words, the requirement is that the rule $q_z$ be of algebraic degree of precision at least $2n$.

Example 6.1. Here we choose $q_z$, the quadrature rule over $z$, to be the $(n+1)$-point Gauss-Legendre rule. Then (6.7) is satisfied, because this rule has degree of precision $2n+1$. This choice gives $m = 2(n+1)^2$ for the total number of points. (Stroud [19] gives analogous tensor product Gauss rules of specified precision for the sphere for all $r \geqslant 3$.) The hyperinterpolation approximation obtained with this rule has been studied in [10], and the non-optimal result $\|L_n\|_{C \to C} \leqslant cn^2$ is proved there.

EXAMPLE 6.2. Next choose $q_z$ to be the Clenshaw–Curtis rule [4],

$$q_z h = \sum_{j=0}^{2n} v_j h\left(\cos \frac{j\pi}{2n}\right).$$

This is an interpolatory rule, in which (as pointed out by Imhof [12]), the weights can be written explicitly as

$$v_0 = v_{2n} = \frac{1}{n} \sum_{k=0}^{n}{}'' \frac{1}{1-4k^2} \tag{6.8a}$$

$$v_j = v_{2n-j} = \frac{2}{n} \sum_{k=0}^{n}{}'' \frac{1}{1-4k^2} \cos \frac{kj\pi}{n}, \qquad j = 1, ..., n. \tag{6.8b}$$

(The double prime on the sum indicates that the first and last terms are to be halved.)

This is a positive-weight rule (see [12]) with degree of precision $2n+1$. The resulting value of $m$ is

$$m = (2n-1)\, 2(n+1) + 2 = 4n(n+\tfrac{1}{2}) = 4n^2 + 2n,$$

where we have taken account of the fact that on the sphere there is only one point with $z = +1$, and one with $z = -1$.

EXAMPLE 6.3. In 1933 Fejér [8] discussed an interpolatory quadrature formula based on the "Filippi" points, which are the Clenshaw–Curtis quadrature points excluding the two end-points. The rule was rediscovered recently by [7, Section 4]. The Fejér rule of the appropriate precision is

$$q_z h = \sum_{j=1}^{2n+1} v_j\, h\left(\cos \frac{j\pi}{2n+2}\right), \tag{6.9}$$

where

$$v_j = \frac{2}{n+1} \sin\left(\frac{j\pi}{2n+2}\right) \sum_{\ell=1}^{n+1} \frac{1}{2\ell-1} \sin\left(\frac{(2\ell-1)\, j\pi}{2n+2}\right), \qquad j = 1, ..., 2n+1.$$

It was shown by [8] that $v_j > 0$ for $j = 1, ..., 2n+1$. The hyperinterpolation approximation with this choice of quadrature rule was in effect discussed by [13], and $\|L_n\|_{C \to C}$ shown there to be of order $O(n^{1/2})$. With this rule the value of $m$ is

$$m = (2n+1)(2n+2) = 4n^2 + 6n + 2.$$

Each of the quadrature rules in the three examples above has precision at least $2n$, and therefore generates a valid hyperinterpolation approximation, when combined with the rectangle rule (6.6). It remains to show that these tensor-product rules satisfy the quadrature regularity assumption. The following theorem states a sufficient condition for this property to hold, which is general enough to include all three examples.

THEOREM 6.6.2. *For given $n \geqslant 0$, let $q_\phi q_z$ be a tensor-product rule, with $q_\phi$ given by* (6.6) *and $q_z$ by the positive weight rule*

$$q_z h = \sum_{j=1}^{J} v_j h(z_j),$$

*with $1 \geqslant z_1 > z_2 > \cdots > z_J \geqslant -1$, and $J \geqslant 4$. With $\cos\theta_j := z_j$, $0 \leqslant \theta_j \leqslant \pi$, the quadrature regularity assumption holds for these rules if the following properties all hold for some positive constants $a_0, a_1, a_2, a_3, a_4$:*

$$a_0(n+1) \geqslant J \geqslant a_1(n+1), \tag{6.10a}$$

$$0 < v_j \leqslant a_2 \frac{\sin\theta_j}{n+1} + \frac{a_3}{(n+1)^2}, \qquad j = 1, ..., J, \tag{6.10b}$$

$$\theta_{j+1} - \theta_j \geqslant \frac{a_4}{n+1}, \qquad j = 1, ..., J-1. \tag{6.10c}$$

*Proof.* Note first that $2(n+1)(J-2) + 2 \leqslant m \leqslant 2(n+1)J$, and that in consequence $a_1(n+1)^2 \leqslant m \leqslant 2a_0(n+1)^2$ for $J \geqslant 4$.

Let $A$ be a spherical cap of spherical radius $1/\sqrt{m}$ and axis $(\theta_0, \phi_0)$. Our aim is to prove (5.6) for some constant $c$. It follows from assumptions (6.10c) and (6.10a) that only a bounded number of $j$-values can contribute to the sum of the weights in (5.6): the number of contributing $j$-values is bounded above by $\lfloor (2/\sqrt{m})/(a_4/(n+1)) \rfloor + 1 \leqslant \lfloor 2/(a_4\sqrt{a_1}) \rfloor + 1 =: a_5$ if $J \geqslant 4$. Noting that it is sufficient to establish the quadrature regularity property for each term of the bound in (6.10b) taken separately, we consider first the case $v_j \leqslant a_3/(n+1)^2$. In this case the sum of weights $\mu_k v_j = \pi v_j/(n+1)$ from quadrature points in $A$ is bounded by $2\pi a_5 a_3/(n+1)^2 \leqslant 4\pi a_0 a_3 a_5/m$, so that for this term the quadrature regularity assumption holds.

Now suppose that $v_j \leqslant a_2 \sin\theta_j/(n+1)$. Since $\mu_k = \pi/(n+1)$ for $k=0, ..., 2n+1$, the spherical cap assumption holds in this case if we can show, for each value of $j$ that contributes to the sum of the weights over the spherical cap, that the number $n_j$ of contributing $k$ values (i.e. $n_j := \#\{(\theta_j, \phi_k) \in A, k=0, ..., 2n+1\}$) satisfies

$$n_j \leqslant a_6/\sin\theta_j \tag{6.11}$$

for some constant $a_6 > 0$. This is because in this case the bound for the sum of the quadrature weights is

$$\sum_{(\theta_j, \phi_k) \in A} \mu_k v_j \leqslant \sum_{j=1}^{J} n_j \frac{\pi}{n+1} v_j \leqslant \sum_{j=1}^{J} \frac{a_6}{\sin\theta_j} \frac{\pi}{n+1} \frac{a_2 \sin\theta_j}{n+1} \leqslant \frac{2\pi a_0 a_2 a_5 a_6}{m}.$$

In order to prove (6.11), observe that the constant distance between the points $(\theta_j, \phi_k)$ and $(\theta_j, \phi_{k+1})$ measured along the latitude $\theta = \theta_j$ is $\pi \sin\theta_j/(n+1)$, whereas the total length of the intersection of that latitude with $A$ is bounded above by the circumference of $A$, and hence by $2\pi/\sqrt{m}$. It follows that

$$n_j \leqslant \frac{2\pi/\sqrt{m}}{\pi \sin\theta_j/(n+1)} + 1 \leqslant \frac{a_6}{\sin\theta_j},$$

with $a_6 = 1 + 2/\sqrt{a_1}$. This proves (6.11), completing the proof. ∎

We now show that all three of the Examples above satisfy the conditions of the theorem.

COROLLARY 6.6.3 *Each of Examples* 1, 2 *and* 3 *satisfies the quadrature regularity assumption.*

*Proof.* We show that the conditions (6.10) of Theorem 6.6.2 are satisfied in each case. We begin with Example 6.3. Note first that the weight $v_j$ in the rule (6.9) is bounded by

$$|v_j| \leqslant \frac{c}{n+1} \sin\left(\frac{j\pi}{2n+2}\right) = \frac{c}{n+1} \sin\theta_j, \qquad j=1, ..., 2n+1, \tag{6.12}$$

where $\theta_j = j\pi/(2n+2)$. This follows from the fact that $(4/\pi) \sum_{\ell \geqslant 1} (2\ell-1)^{-1} \sin(2\ell-1)\,\theta$ is the Fourier series of the $2\pi$-periodic function whose value is 1 on $(0, \pi)$ and $-1$ on $(-\pi, 0)$. Although the partial sums exhibit the Gibbs phenomenon, their uniform norms are well known to be uniformly bounded.

Since $J = 2n+2$ and $\theta_j = \pi/(2n+2)$, it is now clear that the conditions of Theorem 6.6.2 are satisfied, with $a_3 = 0$, $a_0 = a_1 = 2$ and $a_4 = \pi/2$.

Now consider the case of Example 6.2, the Clenshaw–Curtis rule. With $\theta_j$ now defined by

$$\theta_j = \frac{j\pi}{2n}, \qquad j = 0, ..., 2n,$$

it follows from (6.8) that the Clenshaw–Curtis weights can be written as

$$v_j = \frac{2}{n}\,\alpha_j \sum_{k=0}^{n}{}'' \frac{1}{1-4k^2} \cos 2k\theta_j, \qquad j = 0, ..., 2n,$$

with $\alpha_0 = \alpha_{2n} = 1/2$ and $\alpha_j = 1$, $j = 1, ..., 2n-1$. Now we observe that the Fourier series of the even, $2\pi$-periodic function $|\sin \theta|$ is

$$|\sin \theta| = \frac{4}{\pi} \sum_{k=0}^{\infty}{}' \frac{1}{1-4k^2} \cos 2k\theta,$$

where the single prime indicates that the first term is to be halved. This allows the weights $v_j$ to be written in the form

$$v_j = \frac{\pi}{2n}\,\alpha_j \sin \theta_j - r_j, \qquad j = 0, ..., 2n, \tag{6.13}$$

where

$$r_j = \frac{2}{n}\,\alpha_j \sum_{k=n}^{\infty}{}' \frac{1}{4k^2-1} \cos \theta_j .$$

The second term of (6.13), being an absolutely convergent Fourier series, can be bounded by

$$|r_j| \leqslant \frac{1}{2n} \sum_{k=n}^{\infty} \frac{1}{(k-1)^2} \leqslant \frac{1}{2n} \int_{n-2}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{2n(n-2)} \leqslant \frac{8}{3(n+1)^2} \qquad \text{if} \quad n \geqslant 3.$$

It is clear from this bound and (6.13) that the three conditions of Theorem 6.6.2 are satisfied. This completes the proof for Example 6.2.

Finally, we turn to Example 6.1 and the Gauss rule. In this case $J = n+1$, and according to Szegö [20, Equation (5.3.14)],

$$v_j \leqslant c\,\frac{\theta_j}{n+1}, \qquad j = 1, ..., \left\lfloor \frac{n+1}{2} \right\rfloor,$$

from which it follows, using $\theta \leqslant \pi/2 \sin\theta$ for $0 \leqslant \theta \leqslant \pi/2$, together with $v_j = v_{n+2-j}$, $\theta_j = \pi - \theta_{n+2-j}$ for $j = 1, ..., n+1$ that

$$v_j \leqslant c\,\frac{\sin\theta_j}{n+1}, \qquad j = 1, ..., n+1.$$

Finally it is known (see [20, Theorem 6.21.2]) that

$$\frac{2j-1}{2n+3}\,\pi \leqslant \theta_j \leqslant \frac{2j}{2n+3}\,\pi, \qquad j = 1, ..., n+1,$$

from which follows

$$\theta_{j+1} - \theta_j \geqslant \frac{\pi}{2n+3}, \qquad j = 1, ..., n,$$

completing the proof. ∎

Before concluding this section, we note one class of tensor-product rule for which the validity of the quadrature-regularity property is open. These are the tensor-product "spherical $t$-designs" of Bajnok [1], which are rules of the form (6.4), (6.5), (6.6), and with the rule $q_z$ of the equal weight form

$$q_z h = \frac{2}{J} \sum_{j=1}^{J} h(z_j).$$

Bajnok proves the existence of rules of this form that are exact for all $h \in \mathbb{P}_{2n}$ and for all $J$ sufficiently large: specifically, he shows existence for

$$J \geqslant 2(2n)^2\,(2n+1)\,\sqrt{(2n+2)(4n+1)(4n+2\,\sqrt{4n+4})} + 5.$$

Note that $J$ is very large, the right hand side being of order $O(n^{4.5})$, thus Theorem 6.6.2 is not available. The practical usefulness of rules with such a large number of points must be questionable.

## 7. INTERPOLATION IN THE UNIFORM NORM

For completeness we briefly review known results for the interpolatory approximation $\Lambda_n f$ in the uniform norm.

Given a fundamental system $\{x_1, ..., x_{d_n}\}$, the Lagrange polynomials $\{\ell_1, ..., \ell_{d_n}\} \subseteq \mathbb{P}_n$ are defined, as usual, by

$$\ell_j \in \mathbb{P}_n, \qquad \ell_j(x_i) = \delta_{ji}, \qquad i, j = 1, ..., d_n. \tag{7.1}$$

For given $f \in C(S^{r-1})$ the classical expression for $\Lambda_n f$ is then

$$\Lambda_n f = \sum_{j=1}^{d_n} f(x_j)\, \ell_j, \tag{7.2}$$

which manifestly satisfies the interpolatory property (3.1). From this it follows easily that

$$\|\Lambda_n\|_{C \to C} = \max_{x \in S^{r-1}} \sum_{j=1}^{d_n} |\ell_j(x)|, \tag{7.3}$$

which is the so-called Lebesgue constant for interpolation.

The value of $\|\Lambda_n\|_{C \to C}$ depends on the choice of the fundamental system $\{x_1, ..., x_{d_n}\}$. One knows that $\|\Lambda_n\|_{C \to C}$ can be made arbitrarily large if the fundamental system is badly chosen. The interesting question is how small $\|\Lambda_n\|_{C \to C}$ can be made by a judicious choice of fundamental system. Little is known about this question.

One known result, from the work of Reimer [16], is that there exists a fundamental system with $\|\Lambda_n\|_{C \to C} \leqslant d_n$. This property holds if the fundamental system is "extremal". A fundamental system $X = \{x_1, ..., x_{d_n}\}$ is extremal if it maximizes $|\det S(x_1, ..., x_{d_n})|$, where

$$S(x_1, ..., x_{d_n}) = \begin{bmatrix} s_1(x_1) & \cdots & s_1(x_{d_n}) \\ \vdots & \ddots & \vdots \\ s_{d_n}(x_1) & \cdots & s_{d_n}(x_{d_n}) \end{bmatrix},$$

and $\{s_1, ..., s_{d_n}\}$ is any fixed basis for $\mathbb{P}_n$. The significance of the fundamental system being extremal follows from the explicit representation for $\ell_j$,

$$\ell_j(x) = \frac{\det S(x_1, ..., x_{j-1}, x, x_{j+1}, ..., x_{d_n})}{\det S(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_{d_n})}, \qquad j = 1, ..., d_n, \quad x \in S^{r-1},$$

in that the extremal property gives immediately

$$\|\ell_j\|_\infty = 1, \qquad j = 1, ..., d_n,$$

and hence from (7.3)

$$\|\Lambda_n\|_{C \to C} \leqslant d_n.$$

This bound, which for $r = 3$ gives $\|\Lambda_n\|_{C \to C} \leqslant (n+1)^2$, is sometimes very pessimistic.

Reimer [15] and Freeden *et al.* [9] have pointed out that the "Lagrangian square sums" can play a useful role in the estimation of

$\|\Lambda_n\|_{C\to C}$, in the following way. By an application of the Cauchy–Schwarz inequality to (7.3), we obtain

$$\|\Lambda_n\|_{C\to C} \leqslant d_n^{1/2} \max_{x\in S^{r-1}} \left(\sum_{j=1}^{d_n} \ell_j(x)^2\right)^{1/2}. \tag{7.4}$$

Now the matrix $G$ with elements defined by (4.3) and (4.12) is a symmetric positive-definite matrix, with eigenvalues $\lambda_j$ which can be chosen to satisfy

$$0 < \lambda_{\min} = \lambda_1 \leqslant \cdots \leqslant \lambda_{d_n} = \lambda_{\max}.$$

It is well known that upper and lower bounds on the Lagrangian square sums are given by

$$\frac{G_n(x, x)}{\lambda_{\max}} \leqslant \sum_{j=1}^{d_n} \ell_j(x)^2 \leqslant \frac{G_n(x, x)}{\lambda_{\min}},$$

which is Theorem 1 of Reimer [15] or Lemma 7.2.1 of Freeden *et al.* [9]. Noting that $G_n(x, x)$ is independent of $x$, it is useful to rewrite this as

$$\frac{\lambda_{\text{avg}}}{\lambda_{\max}} \leqslant \sum_{j=1}^{d_n} \ell_j(x)^2 \leqslant \frac{\lambda_{\text{avg}}}{\lambda_{\min}}, \tag{7.5}$$

where by (4.8),

$$\lambda_{\text{avg}} := \frac{\lambda_1 + \cdots + \lambda_{d_n}}{d_n} = \frac{\operatorname{Tr} G}{d_n} = \widetilde{G}_n(1) = \frac{d_n}{|S^{r-1}|}, \tag{7.6}$$

since all diagonal elements have the same value $\widetilde{G}_n(1)$. Using (7.5) together with (7.4) now gives

$$\|\Lambda_n\|_{C\to C} \leqslant d_n^{1/2} \left(\frac{\lambda_{\text{avg}}}{\lambda_{\min}}\right)^{1/2}, \tag{7.7}$$

which is Corollary 2 of [15] and Lemma 7.2.2 of [9].

Reimer notes in particular (in Corollary 3 of [15]) that in the special case that the eigenvalues are all equal, i.e. $\lambda_1 = \cdots = \lambda_{d_n}$, the result reduces to

$$\|\Lambda_n\|_{C\to C} \leqslant d_n^{1/2}. \tag{7.8}$$

This result is also given by Theorem 5.5.2 for the hyperinterpolation operator, if we use the fact (see [18]) that interpolation is a special case of hyperinterpolation when the eigenvalues are all equal. We note, though, that for $r \geqslant 3$ and $n \geqslant 3$ the eigenvalues can **not** be equal. This follows (as pointed out by [16]) from the fact that equality of the eigenvalues would imply, via (7.5), that $\sum \ell_j(x)^2 \equiv 1$ for $x \in S^{r-1}$, yet the latter is shown by Bos [3] to be impossible for $r \geqslant 3$ and $n \geqslant 3$.

At the present time it is an open question whether the bound (7.7) can be improved. In the case $r = 3$ the bound is $(n+1)(\lambda_{\text{avg}}/\lambda_{\text{min}})^{1/2} \geqslant n+1$. In a future paper we shall report empirical indications of the existence of fundamental systems that give a rate of growth for the norm $\|\Lambda_n\|$ close to $O(n)$.

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